Four-loop anomalous dimensions for radiative flavour-changing decays

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ABSTRACT: We evaluate the complete four-loop anomalous dimension matrix that is necessary for determining the effective flavour-changing neutral current couplings $\bar{q}q'\gamma$ and $\bar{q}q'g$ at the next-to-next-to-leading order in QCD. The resulting $\mathcal{O}(\alpha_s^2(\mu_b))$ correction to the $\bar{B} \to X_s\gamma$ branching ratio amounts to around -2.9% for $\mu_b = 5\,\text{GeV}$, and -4.4% for $\mu_b = 2.5\,\text{GeV}$.

Contents

1. Introduction

Flavour-Changing Neutral Current (FCNC) decays are well known to provide stringent constraints on extensions of the Standard Model (SM) because their SM amplitudes are loop-suppressed. An additional chirality suppression factor m_q/M_W for radiative FCNC transitions $q \to q'\gamma$ makes these processes particularly attractive. The chirality suppression may be off-set in certain new-physics models like the Minimal Supersymmetric Standard Model with large $\tan\beta$ [1] or left-right models [2]. Precise SM calculations of the radiative flavour-changing decay rates are thus vital for our ability to derive constraints on new physics from these observables.

In the present paper, we evaluate the complete four-loop Anomalous Dimension Matrix (ADM) that is necessary for calculating the effective FCNC couplings $\bar{q}q'\gamma$ and $\bar{q}q'g$ at the Next-to-Next-to-Leading Order (NNLO) in QCD. Our main motivation is contributing to the NNLO calculation of the inclusive $\bar{B} \to X_s\gamma$ branching ratio that is generated at the quark level by the $b \to s\gamma$ transition. For this decay, both the current experimental errors and the non-perturbative effects are smaller than the perturbative NNLO corrections [3]. Our four-loop $\mathcal{O}(\alpha_s^2(\mu_b))$ contributions turn out to suppress $\mathcal{B}(\bar{B} \to X_s\gamma)$ by around 2.9% for $\mu_b = 5\,\mathrm{GeV}$, and 4.4% for $\mu_b = 2.5\,\mathrm{GeV}$. The dominant part of this effect has already been included in the phenomenological NNLO analysis of refs. [3, 4].

The article is organized as follows. In section 2, we introduce the relevant effective Lagrangian and describe the structure of the Renormalization Group Equations (RGEs) for the Wilson coefficients. In section 3, the bare four-loop calculation is described. Section 4

is devoted to the calculation of subdivergences. Our final results for the renormalization constants and ADMs are given in section 5. The numerical size of the four-loop effects is analyzed in section 6. We summarize in section 7. Appendix A contains solutions to the RGEs for all the Wilson coefficients that matter for $b \to s\gamma$ at the NNLO before including higher-order electroweak corrections.

2. Effective lagrangian

All the FCNC processes that have been observed so far are generated at the electroweak scale $\mu_0 \sim M_W, m_t$, while the relevant momentum transfer scale μ_f is much lower. Large logarithm $\left(\alpha_s \ln \mu_0^2/\mu_f^2\right)^n$ resummation at each order of the perturbation series in $\alpha_s(\mu_f)$ is necessary to obtain viable results. It is achieved by making use of the RGEs in an effective theory that arises from the SM after decoupling of the heavy electroweak bosons and the top quark [5].

The effective Lagrangian that matters for the $q \to q'\gamma$ and $q \to q'g$ transitions at the leading order in the electroweak interactions is given below. For definiteness, we specify the quark flavours as in the $b \to s\gamma$ case.

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD} \times \text{QED}}(u,d,s,c,b) \; + \; \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) Q_i \; + \; \dots \qquad (2.1)$$

Here, G_F and V_{ij} stand for the Fermi coupling constant and the quark mixing matrix elements, respectively. The Wilson coefficients $C_i(\mu)$ play the role of coupling constants at the effective weak vertices (operators) Q_i. The operator basis is chosen as in ref. [6] to avoid difficulties with γ_5 in dimensional regularization.

$$
\begin{aligned}
Q_1 &= (\bar{s}\gamma_\mu T^a P_L c)(\bar{c}\gamma^\mu T^a P_L b), \\
Q_2 &= (\bar{s}\gamma_\mu P_L c)(\bar{c}\gamma^\mu P_L b), \\
Q_3 &= (\bar{s}\gamma_\mu P_L b) \sum_q (\bar{q}\gamma^\mu q), \\
Q_4 &= (\bar{s}\gamma_\mu T^a P_L b) \sum_q (\bar{q}\gamma^\mu T^a q), \\
Q_5 &= (\bar{s}\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} P_L b) \sum_q (\bar{q}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} q), \\
Q_6 &= (\bar{s}\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} T^a P_L b) \sum_q (\bar{q}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} T^a q), \\
Q_7 &= \tfrac{e}{16\pi^2} \left[\bar{s}\sigma^{\mu\nu}(m_s P_L + m_b P_R)b\right] F_{\mu\nu}, \\
Q_8 &= \tfrac{g}{16\pi^2} \left[\bar{s}\sigma^{\mu\nu}(m_s P_L + m_b P_R)T^a b\right] G_{\mu\nu}^a,
\end{aligned}
\qquad (2.2)
$$

where $P_{L,R} = (1 \mp \gamma_5)/2$. For simplicity, terms proportional to the small V_{ub} mixing have been neglected here. The dots on the r.h.s. of eq. (2.1) stand for ultraviolet (UV) counterterms. Apart from $Q_1, \dots, Q_8$, the counterterms contain unphysical operators that matter only off-shell and/or in $D \neq 4$ spacetime dimensions (see section 4).

Instead of the original $C_i(\mu)$, it is more convenient to work with certain linear combinations of them, the so-called "effective" Wilson coefficients

$$
C_i^{\text{eff}}(\mu) = \begin{cases}
C_i(\mu), & \text{for } i = 1,\dots,6, \\
C_7(\mu) + \sum_{j=1}^{6} y_j C_j(\mu), & \text{for } i = 7, \\
C_8(\mu) + \sum_{j=1}^{6} z_j C_j(\mu), & \text{for } i = 8.
\end{cases}
\qquad (2.3)
$$

The numbers y_j and z_j are defined so that the $b \to s\gamma$ and $b \to sg$ amplitudes at the Leading Order (LO) in QCD are proportional to C_7^{eff} and C_8^{eff}, respectively [7]. In the $\overline{\text{MS}}$ scheme with fully anticommuting γ_5 which is used in our calculation, one finds, $\vec{y} = (0, 0, -\frac{1}{3}, -\frac{4}{9}, -\frac{20}{3}, -\frac{80}{9})$ and $\vec{z} = (0, 0, 1, -\frac{1}{6}, 20, -\frac{10}{3})$ [6].

The RGEs for $C_i^{\text{eff}}(\mu)$

$$\mu \frac{d}{d\mu} C_i^{\text{eff}}(\mu) = C_j^{\text{eff}}(\mu) \gamma_{ji}^{\text{eff}}(\mu) \tag{2.4}$$

are governed by the ADM

$$\hat{\gamma}^{\text{eff}}(\mu) = \sum_{n \geq 0} \widetilde{\alpha}^{n+1} \hat{\gamma}^{(n)\text{eff}} \tag{2.5}$$

where $\widetilde{\alpha} = \alpha_s(\mu)/(4\pi)$. This matrix is determined from the effective theory renormalization constants. The matrices $\hat{\gamma}^{(n)\text{eff}}$ have a block-triangular structure

$$\hat{\gamma}^{(n)\text{eff}} = \begin{pmatrix} A_{6\times 6}^{(n)} & B_{6\times 2}^{(n)} \\ 0_{2\times 6} & C_{2\times 2}^{(n)} \end{pmatrix}. \tag{2.6}$$

The down-left block vanishes because the dipole operators Q_7 and Q_8 are actually dimension-five ones, and thus generate no UV divergences in dimension-six four-quark amplitudes. The diagonal blocks $A^{(n)}$ and $C^{(n)}$ are found from $(n+1)$-loop renormalization constants. Non-vanishing contributions to the off-diagonal blocks $B^{(n)}$ arise at two and more loops only. Once the normalization conventions for Q_7 and Q_8 are chosen as in eq. (2.2), the blocks $B^{(n)}$ contain information on $(n+2)$-loop renormalization.[1]

The complete matrices $\hat{\gamma}^{(0)\text{eff}}$ and $\hat{\gamma}^{(1)\text{eff}}$ together with the relevant references can be found in ref. [6]. We quote these results in section 5. The three-loop block $B^{(1)}$ was confirmed in ref. [8]. At the NNLO, one needs to know the full 8×8 matrix $\hat{\gamma}^{(2)\text{eff}}$. The three-loop blocks $A^{(2)}$ and $C^{(2)}$ were calculated in refs. [9] and [10], respectively. In the present paper, we evaluate the four-loop block $B^{(2)}$.

3. Bare four-loop calculation

The renormalization constants contributing to the matrix $B^{(2)}$ are found after subloop renormalization from the UV-divergent parts of four-loop diagrams like the one shown in figure 1. Each of the operators $Q_1, \ldots Q_6$ must be considered in the effective four-quark vertex, and the external gauge boson can be either a photon or a gluon. The overall number of relevant four-loop diagrams turns out to be 21986, when different color and gamma matrix structures are treated together.

In order to simplify the calculation as much as possible, an infrared (IR) rearrangement as in refs. [11–14] has been applied. In this approach, a Taylor expansion in the external momenta is performed after introducing a common mass in all the propagator

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[1]Inverse powers of coupling constants are sometimes used in the dipole operator normalization to make $\hat{\gamma}^{(n)\text{eff}}$ a purely $(n+1)$-loop object. Such a choice is convenient at intermediate steps (see section 5), but the final results for the RGEs are more compact when the normalization as in eq. (2.2) is applied.
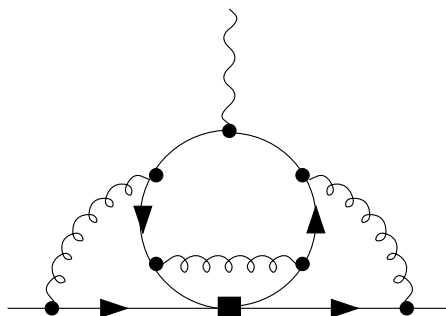
Figure 1: One of the $\mathcal{O}(10^4)$ calculated four-loop diagrams.

denominators. The order of the expansion is determined by the dimensions of the operator insertion and the Green's function. In the present case, up to two derivatives have to be applied. Afterwards, the problem is reduced to evaluation of single-scale fully massive vacuum integrals.

In principle, the contribution of any graph can be mapped onto the following Dirac structures

$$S_j = \left(\gamma_\mu \slashed{p} \slashed{k},\ \gamma_\mu \left(p \cdot k \right),\ \gamma_\mu p^2,\ \gamma_\mu k^2,\ \slashed{p} k_\mu,\ \slashed{p} p_\mu,\ \slashed{k} p_\mu,\ \slashed{k} k_\mu, \right.$$
$$\left. m_b \slashed{k} \gamma_\mu,\ m_b \gamma_\mu \slashed{k},\ m_b \slashed{p} \gamma_\mu,\ m_b \gamma_\mu \slashed{p},\ M^2 \gamma_\mu \right)_j, \qquad (3.1)$$

where p and k stand for the incoming b-quark and the outgoing photon/gluon momenta, respectively, whereas M denotes the regulator mass. However, due to the huge computational resource requirements, only the coefficients at the structures S_j for $j = 7, 9, 10, 11$ have been calculated. Only these four structures are needed to determine the sought ADM $B^{(2)}$ (see section 4).

Comparing the vertex diagram structure and the Taylor expansion depth with those needed in the evaluation of the four-loop β-function with a single gauge parameter, we note that the maximal powers of numerators and denominators in the integrals are the same. This has allowed us to use the solution of the integration-by-parts identities and the master integrals determined in ref. [14]. All the salient details of the employed techniques can be found in that paper.

4. Subdivergences

Considering renormalization of all the operators up to dimension six that can arise in the effective theory (2.1) is necessary for proper subtraction of subdivergences in our calculation. Since the four-loop diagrams are evaluated in the usual Feynman-'t Hooft gauge (without background fields), in $D = 4 - 2\epsilon$ dimensions and off-shell, the following types of counterterms must be taken into account apart from the usual QCD ones:

- the original ("physical") operators from eq. (2.2),

- gauge-invariant operators that vanish by the QCD×QED Equations Of Motion (EOM),

- gauge-variant EOM-vanishing operators,

- BRST-exact operators, i.e. operators that can be written as Becchi-Rouet-Stora-Tyutin variations of other operators, and

- evanescent operators, i.e. operators that vanish in $D = 4$ spacetime dimensions by Dirac-algebra identities, but have to be included for $D \neq 4$.

One begins with writing down all the possible operators of dimension up to six with proper flavour quantum numbers that fall into the above five classes. The potentially infinite set of evanescent objects is reduced to a finite one at each given order of the perturbation series once the prescription of ref. [15] is applied.

The long list of relevant operators can be further reduced by taking into account that the SM Lagrangian is invariant under the following discrete transformations of its fields and parameters (already after the electroweak symmetry breaking and mass matrix diagonalization):

- $b \leftrightarrow s,\ \ m_b \leftrightarrow m_s,\ \ V_{Qb} \leftrightarrow V_{Qs}\ \ (Q = u, c, t)$.

- CP transformation and $V \to V^*$.

A combination of these two transformations maps the $\Delta B = -\Delta S = 1$ interaction terms onto themselves, so it must leave $\mathcal{L}_{\text{eff}}$ (2.1) invariant. The relative plus sign between the m_s and m_b terms in the dipole operators Q_7 and Q_8 is fixed by this very symmetry. Once the operators are determined, terms proportional to m_s are often neglected. Their effect on the $\bar{B} \to X_s \gamma$ branching ratio is negligible ($\mathcal{O}(m_s^2/m_b^2)$) but they matter for the CP asymmetry. Some details on selecting the right number of gauge-invariant operators have been given in section 5.3 of ref. [16].

Explicit expressions for all the operators that satisfy the above-mentioned conditions can be found in refs. [8, 9], so we do not repeat them here. Let us only note that section 2 of ref. [8] contains all the EOM-vanishing operators, including the ones suppressed by the QED coupling e, and the one that gets generated at three loops only (Q_{22}). On the other hand, the only possible BRST-exact operator and all the evanescent operators that we need here are given in section 3 of ref. [9]. The BRST-exact operator remains irrelevant. In particular, we have checked that the three-loop analogue of the two-loop amplitude described in figure 3 of ref. [9] turns out to mysteriously vanish, as well.

Apart from all the "real" off-shell counterterms that have been mentioned so far, we encounter certain spurious ones that arise due to our particular IR rearrangement. The dimension-three gluon-mass-like counterterm was discussed in detail in refs. [11, 12]. Spurious dimension-four flavour-changing operators that are proportional to the IR-regulator mass (see eq. (18) of ref. [8]) are included, too.

The completeness of the full operators basis has been verified in many ways. For instance, since the number of operators contributing at the tree level to the $b \to s\gamma$ transition is smaller than the number of independent Dirac structures S_j (3.1), there exist seven linear

relations between coefficients at these structures in the renormalized amplitude. All these relations together with three analogous ones for $b \to sg$ have been successfully tested up to three loops [6, 8, 9, 17]. In the present calculation, we determine coefficients at four of the structures S_j only, which leaves us with just one linear relation for $b \to s\gamma$ and none for $b \to sg$. Therefore, we essentially rely on the fact that the operator basis is indeed complete.

Once the $(n \leq 3)$-loop diagrams with counterterms are added to the bare four-loop ones, we obtain an expression of the form

$$A_7 \slashed{k} p_\mu + m_b \left(A_9 \slashed{k} \gamma_\mu + A_{10} \gamma_\mu \slashed{k} + A_{11} \slashed{p} \gamma_\mu \right) \tag{4.1}$$

where the coefficients A_i contain UV-poles only. These poles need to be canceled by tree-level counterterms. First, we subtract such tree-level counterterms that originate from products of the usual QCD renormalization constants with the $(n \leq 3)$-loop operator ones, which modifies the coefficients in eq. (4.1). Let us call the new coefficients A_i'. The linear combination

$$\frac{1}{4} A_7' - \frac{1}{2} A_9' + \frac{1}{2} A_{10}' - \frac{1}{2} A_{11}' \tag{4.2}$$

gives us the sought four-loop contributions to the renormalization constants Z_{i7} and Z_{i8} (for $i = 1, \ldots 6$) in the photonic and gluonic cases, respectively. The single consistency relation that is successfully tested in the photonic case reads $A_9' + A_{10}' + A_{11}' = 0$. Proving that the projection (4.2) is the right one requires studying Feynman rules for each operator in the full basis and verifying that only Q_7 or Q_8 can contribute to this particular combination at the tree level.

A very powerful test of the results in eq. (4.1) is provided by the so-called locality constraints. Let us denote the n-loop $1/\epsilon^m$-contribution to A_i by $A_i^{(nm)}$, i.e.

$$A_i = \sum_{n,m=1}^{4} \frac{\mu^{2n\epsilon}}{\epsilon^m} A_i^{(nm)}. \tag{4.3}$$

The quantities $A_i^{(nm)}$ must satisfy many relations to ensure that all the $(\ln^k \mu)/\epsilon^j$ poles cancel out in A_i. Such a cancellation is necessary for the counterterms to remain polynomial in external momenta, i.e. local in the position space. The explicit relations read

$$12 A_i^{(44)} = -3 A_i^{(14)} = 2 A_i^{(24)} = -3 A_i^{(34)} \tag{4.4}$$

$$6 A_i^{(43)} = -A_i^{(23)} - 3 A_i^{(33)} \tag{4.5}$$

$$8 A_i^{(43)} = A_i^{(13)} - 3 A_i^{(33)} \tag{4.6}$$

$$4 A_i^{(42)} = -A_i^{(12)} - 2 A_i^{(22)} - 3 A_i^{(32)}. \tag{4.7}$$

All these relations have been checked to hold in the photonic and gluonic cases for each of the six operator insertions. One should realize that testing the $1/\epsilon^2$ poles in a four-loop amplitude constitutes a much more sensitive cross-check of the calculation than doing the same in a two-loop amplitude. Since A_i^{4m} and A_i^{1m} have been evaluated with the help of different routines written by different authors, we can be practically sure that our calculation is free of errors like missing a diagram or taking it with a wrong overall factor.

5. Renormalization constants and ADMs

The counterterms in the effective Lagrangian that we have calculated can be written in the following form

$$\Delta \mathcal{L}_{\text{eff}}^{\text{count}} = \frac{4 G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{6} C_i \left(Z_{i7} X_7 + Z_{i8} X_8 \right), \tag{5.1}$$

where

$$X_7 = \frac{16\pi^2}{(Z_g g)^2} Z_0 Q_7 \qquad \text{and} \qquad X_8 = \frac{16\pi^2}{(Z_g g)^2} Z_0 Z_g Z_G^{1/2} Q_8. \tag{5.2}$$

Here, Z_0, Z_g and Z_G are the usual QCD renormalization constants of the mass term, gauge coupling and the gluon field

$$(m\bar{\psi}\psi)_B = Z_0 m\bar{\psi}\psi, \qquad g_B = Z_g g, \qquad G_B^{a\,\mu} = Z_G^{1/2} G^{a\,\mu}. \tag{5.3}$$

Below, we give our results for Z_{i7} and Z_{i8} up to four loops. The "trivial" terms proportional to $\ln(4\pi)$ or to the Euler constant γ_E are omitted, as they have no effect on the $\overline{\text{MS}}$ anomalous dimensions. The Riemann zeta function value $\zeta(3) \simeq 1.20206$ is denoted by ζ_3. The number of active flavours in the effective theory and the sum of their charges are denoted by f and $\overline{Q}$, respectively. For Z_{17} and Z_{27}, the charge Q_u of the charm quark is retained arbitrary. The incoming quark charge is set to $-\frac{1}{3}$ in all the expressions for Z_{i7}, which means that they correspond to the down-type quark $\bar{q}q'\gamma$ coupling after substituting $Q_u = \frac{2}{3}$. For the external up-type quark case, one should multiply all the terms in Z_{i7} that come with no charge factor by -2, and then replace Q_u by $Q_d = -\frac{1}{3}$. The same rule applies to the matrices $B^{(n)}$ that are given in eqs. (5.9)–(5.11) at the end of this section. We find

$$Z_{17} = -\frac{\widetilde{\alpha}^2}{\epsilon}\left(\frac{4}{243} + \frac{1}{3}Q_u\right) + \frac{\widetilde{\alpha}^3}{\epsilon}\left\{\frac{1}{\epsilon}\left[-\frac{274}{6561} - \frac{40}{2187}f + \left(\frac{130}{27} - \frac{8}{27}f\right)Q_u\right] + \frac{3239}{6561} - \frac{32}{6561}f - \left(\frac{352}{81}\right.\right.$$
$$\left.\left. -\frac{1}{81}f\right)Q_u\right\} + \frac{\widetilde{\alpha}^4}{\epsilon}\left\{\frac{1}{\epsilon^2}\left[\frac{97508}{59049} + \frac{487}{6561}f - \frac{104}{6561}f^2 - \left(\frac{1556}{27} - \frac{193}{27}f + \frac{2}{9}f^2\right)Q_u - \frac{5}{108}\overline{Q}\right]\right.$$
$$\left. +\frac{1}{\epsilon}\left[-\frac{93353}{19683} + \frac{94513}{236196}f - \frac{100}{19683}f^2 + \left(\frac{47807}{648} - \frac{599}{81}f + \frac{1}{81}f^2\right)Q_u - \frac{245}{972}\overline{Q}\right] - \frac{1385935}{8503056} - \frac{20435}{6561}\zeta_3 \right.$$
$$\left. +\left(\frac{46625}{354294} + \frac{91}{729}\zeta_3\right)f + \frac{70}{19683}f^2 + \left[\frac{14797}{432} + \frac{8381}{162}\zeta_3 + \left(\frac{11969}{5832} - \frac{8}{81}\zeta_3\right)f + \frac{79}{1458}f^2\right]Q_u \right.$$
$$\left. +\left(\frac{3695}{3888} - \frac{25}{54}\zeta_3\right)\overline{Q}\right\},$$

$$Z_{18} = \frac{167}{648}\frac{\widetilde{\alpha}^2}{\epsilon} + \frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(-\frac{6098}{2187} + \frac{1373}{5832}f\right) - \frac{40655}{34992} + \frac{431}{34992}f\right] + \frac{\widetilde{\alpha}^4}{\epsilon}\left[\frac{1}{\epsilon^2}\left(\frac{11512021}{314928} - \frac{313309}{52488}f + \frac{839}{3888}f^2\right)\right.$$
$$\left. +\frac{1}{\epsilon}\left(\frac{7084465}{839808} + \frac{575321}{314928}f + \frac{659}{34992}f^2\right) - \frac{490821325}{22674816} - \frac{368107}{8748}\zeta_3 - \left(\frac{2647291}{944784} + \frac{10381}{3888}\zeta_3\right)f - \frac{1955}{52488}f^2\right],$$

$$Z_{27} = \frac{\widetilde{\alpha}^2}{\epsilon}\left(\frac{8}{81} + 2Q_u\right) + \frac{\widetilde{\alpha}^3}{\epsilon}\left\{\frac{1}{\epsilon}\left[-\frac{4636}{2187} + \frac{80}{729}f - \left(\frac{206}{9} - \frac{16}{9}f\right)Q_u\right] + \frac{6698}{2187} + \frac{64}{2187}f + \left(\frac{128}{27}\right.\right.$$
$$\left.\left. -\frac{2}{27}f\right)Q_u\right\} + \frac{\widetilde{\alpha}^4}{\epsilon}\left\{\frac{1}{\epsilon^2}\left[\frac{560390}{19683} - \frac{7400}{2187}f + \frac{208}{2187}f^2 + \left(\frac{4379}{18} - \frac{323}{9}f + \frac{4}{3}f^2\right)Q_u + \frac{5}{18}\overline{Q}\right]\right.$$
$$\left. +\frac{1}{\epsilon}\left[-\frac{735973}{13122} + \frac{173111}{39366}f + \frac{200}{6561}f^2 - \left(\frac{39427}{216} - \frac{613}{27}f + \frac{2}{27}f^2\right)Q_u + \frac{245}{162}\overline{Q}\right] + \frac{3142663}{1417176} + \frac{5068}{2187}\zeta_3 \right.$$

$$-\left(\frac{79754}{59049}+\frac{146}{243}\zeta_3\right)f-\frac{140}{6561}f^2+\left[\frac{11699}{36}-\frac{6140}{27}\zeta_3-\left(\frac{17153}{972}+\frac{128}{27}\zeta_3\right)f-\frac{79}{243}f^2\right]Q_u$$
$$+\left(-\frac{3695}{648}+\frac{25}{9}\zeta_3\right)\overline{Q}\Big\},$$

$$Z_{28}=\frac{19}{27}\frac{\widetilde{\alpha}^2}{\epsilon}+\frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(-\frac{23063}{2916}+\frac{571}{972}f\right)+\frac{5167}{2916}-\frac{917}{5832}f\right]+\frac{\widetilde{\alpha}^4}{\epsilon}\left[\frac{1}{\epsilon^2}\left(\frac{10632025}{104976}-\frac{63787}{4374}f+\frac{161}{324}f^2\right)\right.$$
$$\left.-\frac{1}{\epsilon}\left(\frac{11481667}{139968}-\frac{714847}{52488}f+\frac{1307}{5832}f^2\right)+\frac{252836197}{3779136}-\frac{487691}{5832}\zeta_3-\left(\frac{6112793}{629856}+\frac{1679}{648}\zeta_3\right)f-\frac{2489}{17496}f^2\right],$$

$$Z_{37}=\frac{16}{81}\frac{\widetilde{\alpha}^2}{\epsilon}+\frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(-\frac{18236}{2187}+\frac{160}{729}f+4\overline{Q}\right)+\frac{49144}{2187}-\frac{1816}{2187}f-\frac{56}{9}\overline{Q}\right]+\frac{\widetilde{\alpha}^4}{\epsilon}\left\{\frac{1}{\epsilon^2}\left[\frac{2436463}{19683}-\frac{27850}{2187}f\right.\right.$$
$$\left.+\frac{416}{2187}f^2-\left(\frac{1265}{18}-\frac{14}{3}f\right)\overline{Q}\right]+\frac{1}{\epsilon}\left[-\frac{10574227}{26244}+\frac{1737559}{39366}f-\frac{5864}{6561}f^2+\left(\frac{76669}{648}-\frac{22}{3}f\right)\overline{Q}\right]$$
$$\left.+\frac{141396541}{708588}-\frac{59074}{2187}\zeta_3-\left(\frac{13789403}{472392}+\frac{598}{243}\zeta_3\right)f-\frac{244}{6561}f^2+\left[\frac{18007}{324}-\frac{56}{3}\zeta_3-\left(\frac{1}{27}+\frac{8}{9}\zeta_3\right)f\right]\overline{Q}\right\},$$

$$Z_{38}=\frac{92}{27}\frac{\widetilde{\alpha}^2}{\epsilon}+\frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(-\frac{63293}{1458}+\frac{4085}{972}f\right)+\frac{87031}{1458}-\frac{3361}{1458}f\right]+\frac{\widetilde{\alpha}^4}{\epsilon}\left[\frac{1}{\epsilon^2}\left(\frac{32831857}{52488}-\frac{3821329}{34992}f+\frac{2891}{648}f^2\right)\right.$$
$$+\frac{1}{\epsilon}\left(-\frac{97744615}{69984}+\frac{73524781}{419904}f-\frac{19453}{5832}f^2\right)+\frac{176999365}{118098}-\frac{1095749}{2916}\zeta_3-\left(\frac{248941103}{5038848}+\frac{505}{162}\zeta_3\right)f$$
$$\left.+\left(\frac{27313}{34992}-\frac{5}{18}\zeta_3\right)f^2\right],$$

$$Z_{47}=-\frac{\widetilde{\alpha}^2}{\epsilon}\left(\frac{170}{243}-\frac{8}{81}f\right)+\frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(\frac{59824}{6561}-\frac{148}{81}f+\frac{80}{729}f^2+\frac{5}{3}\overline{Q}\right)-\frac{104951}{6561}+\frac{2774}{6561}f+\frac{16}{2187}f^2-\frac{70}{27}\overline{Q}\right]$$
$$+\frac{\widetilde{\alpha}^4}{\epsilon}\left\{\frac{1}{\epsilon^2}\left[-\frac{6446822}{59049}+\frac{521993}{19683}f-\frac{17827}{6561}f^2+\frac{208}{2187}f^3-\left(\frac{850}{27}-\frac{20}{9}f\right)\overline{Q}\right]+\frac{1}{\epsilon}\left[\frac{12186259}{39366}-\frac{7760143}{236196}f\right.\right.$$
$$\left.+\frac{58249}{39366}f^2+\frac{56}{6561}f^3+\left(\frac{206399}{3888}-\frac{1175}{324}f\right)\overline{Q}\right]-\frac{895573151}{2125764}-\frac{439534}{6561}\zeta_3+\left(\frac{21721465}{708588}-\frac{8165}{2187}\zeta_3\right)f$$
$$\left.-\left(\frac{71315}{59049}+\frac{160}{243}\zeta_3\right)f^2-\frac{148}{6561}f^3-\left[\frac{42823}{7776}-\frac{1355}{108}\zeta_3-\left(\frac{305}{216}-\frac{25}{27}\zeta_3\right)f\right]\overline{Q}\right\},$$

$$Z_{48}=-\frac{\widetilde{\alpha}^2}{\epsilon}\left(\frac{427}{324}+\frac{37}{216}f\right)+\frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(\frac{84661}{4374}+\frac{71}{36}f-\frac{185}{972}f^2\right)-\frac{540209}{17496}-\frac{40091}{34992}f-\frac{253}{2916}f^2\right]$$
$$+\frac{\widetilde{\alpha}^4}{\epsilon}\left[\frac{1}{\epsilon^2}\left(-\frac{44723201}{157464}-\frac{590221}{34992}f+\frac{216085}{34992}f^2-\frac{259}{1296}f^3\right)+\frac{1}{\epsilon}\left(\frac{238543801}{419904}-\frac{4684067}{314928}f-\frac{1281229}{419904}f^2\right.\right.$$
$$\left.\left.-\frac{253}{1944}f^3\right)-\frac{5633994047}{22674816}+\frac{910901}{4374}\zeta_3+\left(\frac{566791855}{30233088}+\frac{208123}{5832}\zeta_3\right)f+\left(\frac{2176469}{1259712}+\frac{115}{81}\zeta_3\right)f^2+\frac{505}{34992}f^3\right],$$

$$Z_{57}=\frac{464}{81}\frac{\widetilde{\alpha}^2}{\epsilon}+\frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(-\frac{118976}{2187}-\frac{10112}{729}f+40\overline{Q}\right)+\frac{257344}{729}+\frac{16832}{2187}f-\frac{2288}{9}\overline{Q}\right]+\frac{\widetilde{\alpha}^4}{\epsilon}\left\{\frac{1}{\epsilon^2}\left[\frac{28293352}{19683}\right.\right.$$
$$\left.+\frac{201176}{2187}f-\frac{32008}{2187}f^2-\left(\frac{6160}{9}-\frac{140}{3}f\right)\overline{Q}\right]+\frac{1}{\epsilon}\left[-\frac{17496454}{2187}+\frac{3399452}{19683}f+\frac{83368}{6561}f^2+\left(\frac{296492}{81}\right.\right.$$
$$\left.\left.-\frac{752}{3}f\right)\overline{Q}\right]+\frac{1744567561}{177147}+\frac{9416120}{2187}\zeta_3-\left(\frac{59440775}{118098}-\frac{4708}{243}\zeta_3\right)f-\left(\frac{92}{81}-\frac{16}{27}\zeta_3\right)f^2$$
$$\left.-\left[\frac{1095949}{648}+\frac{1210}{9}\zeta_3-\left(\frac{1934}{27}-\frac{80}{9}\zeta_3\right)f\right]\overline{Q}\right\},$$

$$Z_{58}=\frac{\widetilde{\alpha}^2}{\epsilon}\left(\frac{2192}{27}+9f\right)-\frac{\widetilde{\alpha}^3}{\epsilon}\left[\frac{1}{\epsilon}\left(\frac{713164}{729}+\frac{5494}{243}f-8f^2\right)-\frac{452714}{243}-\frac{97043}{1458}f+\frac{7}{3}f^2\right]+\frac{\widetilde{\alpha}^4}{\epsilon}\left[\frac{1}{\epsilon^2}\left(\frac{87867110}{6561}\right.\right.$$
$$\left.-\frac{1639918}{2187}f-\frac{21299}{162}f^2+\frac{43}{6}f^3\right)-\frac{1}{\epsilon}\left(\frac{25933753}{729}-\frac{115452017}{52488}f-\frac{684959}{2916}f^2+\frac{28}{9}f^3\right)$$
$$\left.+\frac{11439938489}{236196}-\frac{11082938}{729}\zeta_3-\left(\frac{269786863}{629856}+\frac{288731}{162}\zeta_3\right)f-\left(\frac{86423}{648}+\frac{635}{9}\zeta_3\right)f^2-\frac{37}{54}f^3\right],$$

$$Z_{67}=-\frac{\widetilde{\alpha}^2}{\epsilon}\left(\frac{3008}{243}-\frac{80}{81}f-12\overline{Q}\right)+\frac{\widetilde{\alpha}^3}{\epsilon}\left\{\frac{1}{\epsilon}\left[\frac{925216}{6561}-\frac{19720}{729}f+\frac{800}{729}f^2+\left(-\frac{404}{3}+\frac{32}{3}f\right)\overline{Q}\right]-\frac{389636}{2187}\right.$$

$$+\frac{154280}{6561}f+\frac{448}{2187}f^2-\left(\frac{1468}{27}+\frac{28}{9}f\right)\overline{Q}\Big\}+\frac{\widetilde{\alpha}^4}{\epsilon}\Big\{\frac{1}{\epsilon^2}\Big[-\frac{82789640}{59049}+\frac{8283332}{19683}f-\frac{234328}{6561}f^2+\frac{2080}{2187}f^3$$

$$+\left(\frac{37826}{27}-\frac{1885}{9}f+8f^2\right)\overline{Q}\Big]+\frac{1}{\epsilon}\Big[\frac{20027615}{6561}-\frac{39955153}{59049}f+\frac{708617}{19683}f^2+\frac{1424}{6561}f^3-\left(\frac{17755}{243}\right.$$

$$\left.-\frac{8992}{81}f+\frac{28}{9}f^2\right)\overline{Q}\Big]-\frac{2247378701}{531441}-\frac{766276}{6561}\zeta_3+\left(\frac{195301451}{354294}-\frac{21056}{2187}\zeta_3\right)f-\left(\frac{1251734}{59049}\right.$$

$$\left.+\frac{1540}{243}\zeta_3\right)f^2-\frac{1432}{6561}f^3+\left[\frac{7375087}{3888}-\frac{40664}{27}\zeta_3-\left(\frac{62269}{324}+\frac{142}{27}\zeta_3\right)f-\frac{74}{81}f^2\right]\overline{Q}\Big\},$$

$$Z_{68}=-\frac{\widetilde{\alpha}^2}{\epsilon}\left(\frac{1906}{81}-\frac{55}{27}f\right)+\frac{\widetilde{\alpha}^3}{\epsilon}\Big[\frac{1}{\epsilon}\left(\frac{873038}{2187}-\frac{9530}{243}f+\frac{695}{486}f^2\right)-\frac{726064}{729}-\frac{410713}{17496}f-\frac{6031}{2916}f^2\Big]$$

$$+\frac{\widetilde{\alpha}^4}{\epsilon}\Big[\frac{1}{\epsilon^2}\left(-\frac{131785021}{19683}+\frac{448189}{486}f-\frac{380411}{8748}f^2+\frac{80}{81}f^3\right)+\frac{1}{\epsilon}\left(\frac{184494545}{8748}-\frac{940982911}{629856}f+\frac{3173417}{104976}f^2\right.$$

$$\left.-\frac{2821}{972}f^3\right)-\frac{97818607289}{5668704}-\frac{4146212}{2187}\zeta_3+\left(\frac{3023834561}{3779136}+\frac{309493}{1458}\zeta_3\right)f-\left(\frac{3278851}{78732}-\frac{175}{324}\zeta_3\right)f^2$$

$$-\frac{997}{4374}f^3\Big].\tag{5.4}$$

The n-loop $1/\epsilon^n$ poles vanish in the above renormalization constants, which is a consequence of the lack of one-loop contributions to Z_{i7} and Z_{i8}.

The operators X_i (5.2) differ by a multiplicative factor of $1/\widetilde{\alpha}$ from the bare versions of Q_7 and Q_8. We have used X_i in the actual renormalization constant calculation. Each n-loop term is then of order $\widetilde{\alpha}^n$, which makes the simple matrix relation between renormalization constants and anomalous dimensions applicable. Such a relation up to three-loops has been given in appendix B of ref. [12]. Extending that result to four loops in the same notation (but with $\kappa^{0n}=0$) yields

$$\hat{\gamma}=2\hat{a}^{11}g^2+g^4\left[4\hat{a}^{12}-2\hat{a}^{01}\hat{a}^{11}-2\hat{a}^{11}\hat{a}^{01}-4\kappa^{11}\hat{a}^{01}\right]+g^6\left[6\hat{a}^{13}-4\hat{a}^{12}\hat{a}^{01}-2\hat{a}^{01}\hat{a}^{12}-4\hat{a}^{02}\hat{a}^{11}\right.$$

$$-2\hat{a}^{11}\hat{a}^{02}+2\hat{a}^{01}\hat{a}^{11}\hat{a}^{01}+2\hat{a}^{11}(\hat{a}^{01})^2+2(\hat{a}^{01})^2\hat{a}^{11}+4\kappa^{11}(\hat{a}^{01})^2-8\kappa^{11}\hat{a}^{02}-8\kappa^{12}\hat{a}^{01}\big]$$

$$+g^8\left[8\hat{a}^{14}-6\hat{a}^{13}\hat{a}^{01}-2\hat{a}^{01}\hat{a}^{13}-6\hat{a}^{03}\hat{a}^{11}-2\hat{a}^{11}\hat{a}^{03}-4\hat{a}^{12}\hat{a}^{02}-4\hat{a}^{02}\hat{a}^{12}+4\hat{a}^{02}\hat{a}^{01}\hat{a}^{11}\right.$$

$$+4\hat{a}^{02}\hat{a}^{11}\hat{a}^{01}+2\hat{a}^{01}\hat{a}^{12}\hat{a}^{01}+2\hat{a}^{11}\hat{a}^{02}\hat{a}^{01}+2\hat{a}^{01}\hat{a}^{02}\hat{a}^{11}+2\hat{a}^{11}\hat{a}^{01}\hat{a}^{02}+2\hat{a}^{01}\hat{a}^{11}\hat{a}^{02}$$

$$+4\hat{a}^{12}(\hat{a}^{01})^2+2(\hat{a}^{01})^2\hat{a}^{12}-2\hat{a}^{01}\hat{a}^{11}(\hat{a}^{01})^2-2(\hat{a}^{01})^2\hat{a}^{11}\hat{a}^{01}-2\hat{a}^{11}(\hat{a}^{01})^3-2(\hat{a}^{01})^3\hat{a}^{11}$$

$$-12\hat{a}^{03}\kappa^{11}-12\hat{a}^{01}\kappa^{13}-16\hat{a}^{02}\kappa^{12}+8\hat{a}^{02}\hat{a}^{01}\kappa^{11}+4\hat{a}^{01}\hat{a}^{02}\kappa^{11}+8(\hat{a}^{01})^2\kappa^{12}-4(\hat{a}^{01})^3\kappa^{11}\big]$$

$$+\mathcal{O}(g^{10}).\tag{5.5}$$

After calculating the ADM $\hat{\gamma}$ from eq. (5.5), we multiply the dipole operators by $\widetilde{\alpha}$, which brings us to the normalization of Q_7 and Q_8 as in eq. (2.2). Next, we pass from $\hat{\gamma}$ to $\hat{\gamma}^{\text{eff}}$, which amounts to performing a simple linear transformation[2] that stems from eq. (2.3).

The obtained complete expressions for $\hat{\gamma}^{(n)\text{eff}}$ ($n=0,1,2$) in the $b\to s\gamma$ case ($f=5$,

---

[2]See eq. (38) of ref. [7].

$\overline{Q} = \frac{1}{3}$, $Q_u = \frac{2}{3}$) are as follows:

$$\hat{\gamma}^{(0)\text{eff}} = \begin{bmatrix} -4 & \frac{8}{3} & 0 & -\frac{2}{9} & 0 & 0 & -\frac{208}{243} & \frac{173}{162} \\[4pt] 12 & 0 & 0 & \frac{4}{3} & 0 & 0 & \frac{416}{81} & \frac{70}{27} \\[4pt] 0 & 0 & 0 & -\frac{52}{3} & 0 & 2 & -\frac{176}{81} & \frac{14}{27} \\[4pt] 0 & 0 & -\frac{40}{9} & -\frac{100}{9} & \frac{4}{9} & \frac{5}{6} & -\frac{152}{243} & -\frac{587}{162} \\[4pt] 0 & 0 & 0 & -\frac{256}{3} & 0 & 20 & -\frac{6272}{81} & \frac{6596}{27} \\[4pt] 0 & 0 & -\frac{256}{9} & \frac{56}{9} & \frac{40}{9} & -\frac{2}{3} & \frac{4624}{243} & \frac{4772}{81} \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} & 0 \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & -\frac{32}{9} & \frac{28}{3} \end{bmatrix} \tag{5.6}$$

$$\hat{\gamma}^{(1)\text{eff}} = \begin{bmatrix} -\frac{355}{9} & -\frac{502}{27} & -\frac{1412}{243} & -\frac{1369}{243} & \frac{134}{243} & -\frac{35}{162} & -\frac{818}{243} & \frac{3779}{324} \\[4pt] -\frac{35}{3} & -\frac{28}{3} & -\frac{416}{81} & \frac{1280}{81} & \frac{56}{81} & \frac{35}{27} & \frac{508}{81} & \frac{1841}{108} \\[4pt] 0 & 0 & -\frac{4468}{81} & -\frac{31469}{81} & \frac{400}{81} & \frac{3373}{108} & \frac{22348}{243} & \frac{10178}{81} \\[4pt] 0 & 0 & -\frac{8158}{243} & -\frac{59399}{243} & \frac{269}{486} & \frac{12899}{648} & -\frac{17584}{243} & -\frac{172471}{648} \\[4pt] 0 & 0 & -\frac{251680}{81} & -\frac{128648}{81} & \frac{23836}{81} & \frac{6106}{27} & \frac{1183696}{729} & \frac{2901296}{243} \\[4pt] 0 & 0 & \frac{58640}{243} & -\frac{26348}{243} & -\frac{14324}{243} & -\frac{2551}{162} & \frac{2480344}{2187} & -\frac{3296257}{729} \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & \frac{4688}{27} & 0 \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2192}{81} & \frac{4063}{27} \end{bmatrix}. \tag{5.7}$$

$$\hat{\gamma}^{(2)\text{eff}} = \begin{bmatrix} -\frac{12773}{18} & \frac{745}{9} & \frac{63187}{13122} & -\frac{981796}{6561} & -\frac{202663}{52488} & -\frac{24973}{69984} & \frac{150994745}{1062882} & -\frac{138745277}{354294} \\[4pt] \frac{1177}{2} & 306 & \frac{110477}{2187} & \frac{133529}{8748} & -\frac{42929}{8748} & \frac{354319}{11664} & \frac{138336202}{177147} & -\frac{83347037}{472392} \\[4pt] 0 & 0 & -\frac{3572528}{2187} & -\frac{58158773}{8748} & \frac{552601}{4374} & \frac{6989171}{11664} & -\frac{58397866}{177147} & \frac{5093967523}{944784} \\[4pt] 0 & 0 & -\frac{1651004}{6561} & -\frac{155405353}{52488} & \frac{1174159}{52488} & \frac{10278809}{34992} & -\frac{5108749081}{2125764} & -\frac{9826683847}{5668704} \\[4pt] 0 & 0 & -\frac{147978032}{2187} & -\frac{168491372}{2187} & \frac{11213042}{2187} & \frac{17850329}{2916} & \frac{5824017302}{177147} & \frac{24198694001}{118098} \\[4pt] 0 & 0 & \frac{136797922}{6561} & -\frac{72614473}{13122} & -\frac{9288181}{6561} & -\frac{16664027}{17496} & \frac{3603565835}{531441} & -\frac{122956397803}{1417176} \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & \frac{179768}{81} & 0 \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & -\frac{23444}{81} & \frac{192137}{81} \end{bmatrix}$$

$$+ \zeta_3 \begin{bmatrix} \frac{1472}{3} & -\frac{4288}{9} & -\frac{1360}{81} & -\frac{776}{81} & \frac{124}{81} & \frac{100}{27} & \frac{1272596}{6561} & -\frac{2373229}{4374} \\[4pt] -2144 & -224 & \frac{2720}{27} & -\frac{2768}{27} & -\frac{248}{27} & -\frac{110}{9} & -\frac{2713672}{2187} & -\frac{682658}{729} \\[4pt] 0 & 0 & -\frac{608}{27} & \frac{61424}{27} & -\frac{496}{27} & \frac{2821}{9} & \frac{3236560}{2187} & -\frac{1051828}{729} \\[4pt] 0 & 0 & \frac{88720}{81} & \frac{54272}{81} & -\frac{9274}{81} & -\frac{3100}{27} & \frac{2007886}{6561} & \frac{4858889}{2187} \\[4pt] 0 & 0 & \frac{87040}{27} & \frac{324416}{27} & -\frac{13984}{27} & -\frac{31420}{9} & \frac{112180720}{2187} & -\frac{119487244}{729} \\[4pt] 0 & 0 & \frac{721408}{81} & -\frac{166432}{81} & -\frac{95032}{81} & -\frac{7552}{27} & \frac{15361912}{6561} & -\frac{71430598}{2187} \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & -\frac{21056}{27} & 0 \\[4pt] 0 & 0 & 0 & 0 & 0 & 0 & \frac{22976}{81} & -\frac{48304}{27} \end{bmatrix}. \tag{5.8}$$

For arbitrary values of f, $\overline{Q}$ and Q_u, the blocks $B^{(n)}$ ($n = 0, 1, 2$) read

$$
B^{(0)} = \begin{bmatrix}
\frac{8}{243} - \frac{4}{3} Q_u & \frac{173}{162} \\[4pt]
-\frac{16}{81} + 8 Q_u & \frac{70}{27} \\[4pt]
-\frac{176}{81} & \frac{14}{27} \\[4pt]
\frac{88}{243} - \frac{16}{81} f & \frac{74}{81} - \frac{49}{54} f \\[4pt]
-\frac{6272}{81} & \frac{1736}{27} + 36 f \\[4pt]
\frac{3136}{243} - \frac{160}{81} f + 48 \overline{Q} & \frac{2372}{81} + \frac{160}{27} f
\end{bmatrix} ,
\tag{5.9}
$$

$$
B^{(1)} = \begin{bmatrix}
\frac{12614}{2187} - \frac{64}{2187} f - \frac{374}{27} Q_u + \frac{2}{27} f Q_u & \frac{65867}{5832} + \frac{431}{5832} f \\[4pt]
-\frac{2332}{729} + \frac{128}{729} f + \frac{136}{9} Q_u - \frac{4}{9} f Q_u & \frac{10577}{486} - \frac{917}{972} f \\[4pt]
\frac{97876}{729} - \frac{4352}{729} f - \frac{112}{3} \overline{Q} & \frac{42524}{243} - \frac{2398}{243} f \\[4pt]
-\frac{70376}{2187} - \frac{15788}{2187} f + \frac{32}{729} f^2 - \frac{140}{9} \overline{Q} & -\frac{159718}{729} - \frac{39719}{5832} f - \frac{253}{486} f^2 \\[4pt]
\frac{1764752}{729} - \frac{65408}{729} f - \frac{3136}{3} \overline{Q} & \frac{2281576}{243} + \frac{140954}{243} f - 14 f^2 \\[4pt]
\frac{4193840}{2187} - \frac{324128}{2187} f + \frac{896}{729} f^2 - \frac{1136}{9} \overline{Q} - \frac{56}{3} f \overline{Q} & -\frac{3031517}{729} - \frac{15431}{1458} f - \frac{6031}{486} f^2
\end{bmatrix} ,
\tag{5.10}
$$

$$
B^{(2)} = \begin{bmatrix}
\frac{77506102}{531441} - \frac{875374}{177147} f + \frac{560}{19683} f^2 - \frac{9731}{162} Q_u + \frac{11045}{729} f Q_u + \frac{316}{729} f^2 Q_u + \frac{3695}{486} \overline{Q} \\[4pt]
-\frac{15463055}{177147} + \frac{242204}{59049} f - \frac{1120}{6561} f^2 + \frac{55748}{27} Q_u - \frac{33970}{243} f Q_u - \frac{632}{243} f^2 Q_u - \frac{3695}{81} \overline{Q} \\[4pt]
\frac{102439553}{177147} - \frac{12273398}{59049} f + \frac{5824}{6561} f^2 + \frac{26639}{81} \overline{Q} - \frac{8}{27} f \overline{Q} \\[4pt]
-\frac{2493414077}{1062882} - \frac{9901031}{354294} f + \frac{243872}{59049} f^2 - \frac{1184}{6561} f^3 - \frac{49993}{972} \overline{Q} + \frac{305}{27} f \overline{Q} \\[4pt]
\frac{8808397748}{177147} - \frac{174839456}{59049} f + \frac{1600}{729} f^2 - \frac{669694}{81} \overline{Q} + \frac{10672}{27} f \overline{Q} \\[4pt]
\frac{7684242746}{531441} - \frac{351775414}{177147} f - \frac{479776}{59049} f^2 - \frac{11456}{6561} f^3 + \frac{3950201}{243} \overline{Q} - \frac{130538}{81} f \overline{Q} - \frac{592}{81} f^2 \overline{Q}
\end{bmatrix}
$$

$$
\begin{bmatrix}
-\frac{421272953}{1417176} - \frac{8210077}{472392} f - \frac{1955}{6561} f^2 \\[4pt]
\frac{98548513}{472392} - \frac{5615165}{78732} f - \frac{2489}{2187} f^2 \\[4pt]
\frac{3205172129}{472392} - \frac{108963529}{314928} f + \frac{58903}{4374} f^2 \\[4pt]
-\frac{6678822461}{2834352} + \frac{127999025}{1889568} f + \frac{1699073}{157464} f^2 + \frac{505}{4374} f^3 \\[4pt]
\frac{29013624461}{118098} - \frac{64260772}{19683} f - \frac{230962}{243} f^2 - \frac{148}{27} f^3 \\[4pt]
-\frac{72810260309}{708588} + \frac{2545824851}{472392} f - \frac{33778271}{78732} f^2 - \frac{3988}{2187} f^3
\end{bmatrix}
$$

$$
+ \zeta_3 \begin{bmatrix}
-\frac{112216}{6561} + \frac{728}{729} f + \frac{25508}{81} Q_u - \frac{64}{81} f Q_u - \frac{100}{27} \overline{Q} & -\frac{953042}{2187} - \frac{10381}{486} f \\[4pt]
\frac{365696}{2187} - \frac{1168}{243} f - \frac{51232}{27} Q_u - \frac{1024}{27} f Q_u + \frac{200}{9} \overline{Q} & -\frac{607103}{729} - \frac{1679}{81} f \\[4pt]
\frac{3508864}{2187} - \frac{1904}{243} f - \frac{1984}{9} \overline{Q} - \frac{64}{9} f \overline{Q} & -\frac{1597588}{729} + \frac{13028}{81} f - \frac{20}{9} f^2 \\[4pt]
-\frac{1922264}{6561} + \frac{308648}{2187} f - \frac{1280}{243} f^2 + \frac{1010}{9} \overline{Q} - \frac{200}{27} f \overline{Q} & \frac{2312684}{2187} + \frac{128347}{729} f + \frac{920}{81} f^2 \\[4pt]
\frac{123543040}{2187} - \frac{207712}{243} f + \frac{128}{27} f^2 - \frac{24880}{9} \overline{Q} - \frac{640}{9} f \overline{Q} & -\frac{69359224}{729} - \frac{885356}{81} f - \frac{5080}{9} f^2 \\[4pt]
\frac{7699264}{6561} + \frac{2854976}{2187} f - \frac{12320}{243} f^2 - \frac{108584}{9} \overline{Q} - \frac{1136}{27} f \overline{Q} & -\frac{61384768}{2187} - \frac{685472}{729} f + \frac{350}{81} f^2
\end{bmatrix}
\tag{5.11}
$$

i	1	2	3	4	5	6	7	8
a_i	$\frac{14}{23}$	$\frac{16}{23}$	$\frac{6}{23}$	$-\frac{12}{23}$	0.4086	-0.4230	-0.8994	0.1456
$h_i^{(0)}$	2.2996	-1.0880	-0.4286	-0.0714	-0.6494	-0.0380	-0.0185	-0.0057
$h_i^{(1)}$	0.9075	-0.7550	0.6971	0.0166	-0.7345	0.1784	-0.2608	-0.0493
$h_i^{(2)}$	-13.9117	11.4063	3.4986	-0.0619	-2.4429	2.1210	-0.7558	0.1464

Table 1: The numbers a_i and $h_i^{(n)}$ that occur in eqs. (6.6) and (6.8).

The f- and charge-dependence of the blocks $C^{(n)}$ for $n = 0, 1, 2$ can be found in eqs. (9)–(11) of ref. [10]. The blocks $A^{(n)}$ are charge-independent, while their f-dependence is given in eqs. (36)–(38) of ref. [9].

6. Numerical analysis

Let $\mathcal{B}_{\mathrm{NNLO}}$ denote the NNLO value of the $\bar{B} \to X_s \gamma$ branching ratio, and the superscript "4L→0" indicate that the four-loop matrix $B^{(2)}$ is set to zero in the calculation of a given quantity. Normalizing the difference between $\mathcal{B}_{\mathrm{NNLO}}$ and $\mathcal{B}_{\mathrm{NNLO}}^{\mathrm{4L}\to 0}$ to the LO result, one finds

$$\frac{\mathcal{B}_{\mathrm{NNLO}} - \mathcal{B}_{\mathrm{NNLO}}^{\mathrm{4L}\to 0}}{\mathcal{B}_{\mathrm{LO}}} = \left(\frac{\alpha_s(\mu_b)}{\pi} \right)^2 \frac{C_7^{(2)\mathrm{eff}}(\mu_b) - \left[C_7^{(2)\mathrm{eff}}(\mu_b) \right]^{\mathrm{4L}\to 0}}{8\, C_7^{(0)\mathrm{eff}}(\mu_b)}, \tag{6.1}$$

where μ_b is the low-energy scale at which the RGEs evolution is terminated, while $C_7^{(n)\mathrm{eff}}$ are defined by the perturbative expansion

$$C_7^{\mathrm{eff}}(\mu) = \sum_{n \geq 0} \widetilde{\alpha}^n(\mu) C_7^{(n)\mathrm{eff}}(\mu). \tag{6.2}$$

To evaluate the r.h.s. of eq. (6.1), we need to know the non-vanishing LO initial conditions for the Wilson coefficients at the electroweak scale μ_0. They read [19]
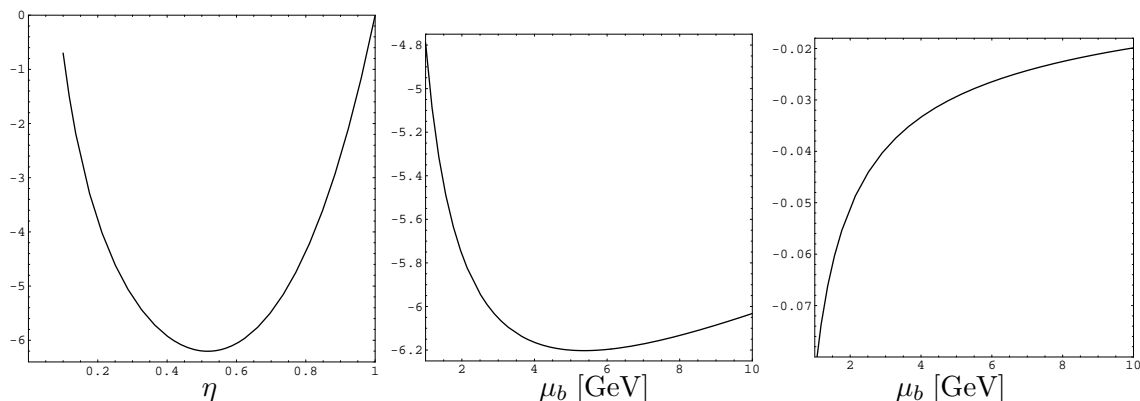
$$C_2^{(0)}(\mu_0) = 1, \tag{6.3}$$

$$C_7^{(0)}(\mu_0) = \frac{3x^3 - 2x^2}{4(x-1)^4} \ln x + \frac{-8x^3 - 5x^2 + 7x}{24(x-1)^3}, \tag{6.4}$$

$$C_8^{(0)}(\mu_0) = \frac{-3x^2}{4(x-1)^4} \ln x + \frac{-x^3 + 5x^2 + 2x}{8(x-1)^3}, \tag{6.5}$$

where $x = m_t^2(\mu_0)/M_W^2$. Substituting them into the analytical solution to the RGEs (see, e.g., section 3.3 of ref. [18]), one obtains

$$\frac{C_7^{(2)\mathrm{eff}}(\mu_b) - \left[C_7^{(2)\mathrm{eff}}(\mu_b) \right]^{\mathrm{4L}\to 0}}{8\, C_7^{(0)\mathrm{eff}}(\mu_b)} = \frac{h_1^{(2)} \eta^{a_1+2} + h_2^{(2)} \eta^{a_2+2} + \sum_{i=3}^{8} h_i^{(2)} \eta^{a_i}}{\eta^{a_2} C_7^{(0)}(\mu_0) + \frac{8}{3}\left(\eta^{a_1} - \eta^{a_2} \right) C_8^{(0)}(\mu_0) + \sum_{i=1}^{8} h_i^{(0)} \eta^{a_i}}, \tag{6.6}$$

Figure 2: The r.h.s. of eq. (6.6) as a function of η (left plot) and μ_b (middle plot). The right plot shows the relative NNLO correction (6.1) to the branching ratio as a function of μ_b.
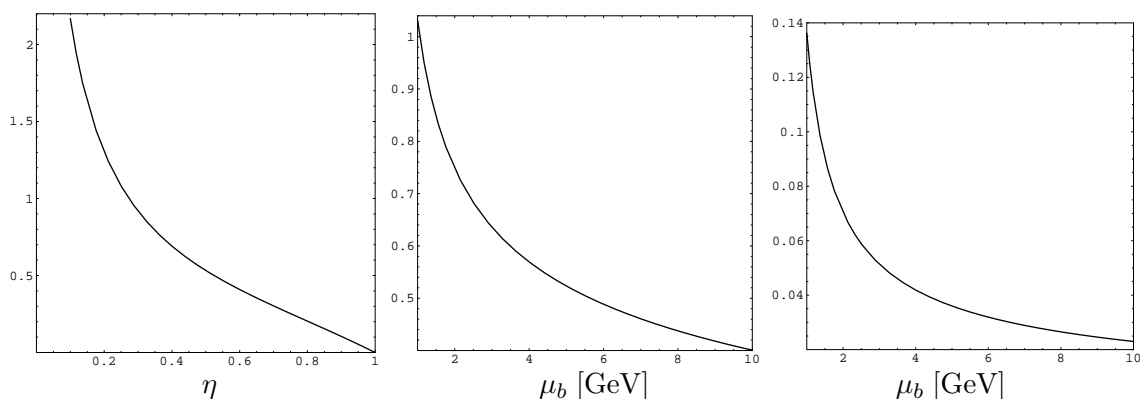
Figure 3: The r.h.s. of eq. (6.8) as a function of η (left plot) and μ_b (middle plot). The right plot shows the relative NLO correction (6.7) to the branching ratio as a function of μ_b.

where $\eta = \alpha_s(\mu_0)/\alpha_s(\mu_b)$. For the numerical evaluation of η, we always use the four-loop RGE for $\alpha_s(\mu)$. The numbers a_i and $h_i^{(n)}$ are given in table 1. They are found from the diagonalization of $\hat{\gamma}^{(0)\mathrm{eff}}$ (5.6) as well as the elements of $\hat{\gamma}^{(1)\mathrm{eff}}$ (5.7) and $\hat{\gamma}^{(2)\mathrm{eff}}$ (5.8). Although $\hat{\gamma}^{(0)\mathrm{eff}}$ can be diagonalized analytically, the decimal approximations presented in table 1 are much more convenient to use. Note that $\sum_{i=1}^{8} h_i^{(n)} = 0$, which follows from the initial condition for C_7 at $\eta = 1$ and from the off-diagonal position of $B^{(n)}$ in $\hat{\gamma}^{(n)\mathrm{eff}}$.

The r.h.s. of eq. (6.6) is shown in the left plot of figure 2 as a function of η for $x = (162/80.4)^2$. The middle plot in the same figure presents this quantity as a function of μ_b for $\mu_0 = 160$ GeV and for the remaining input parameters as listed in appendix A of ref. [4]. The μ_b-dependence of the complete correction (6.1) is shown for the same parameters in the right plot of figure 2.

It is interesting to compare the four-loop effect to the analogous three-loop one at the Next-to-Leading Order (NLO). Let the superscript "$3L \to 0$" indicate neglecting the

three-loop matrix $B^{(1)}$. Then

$$\frac{\mathcal{B}_{\text{NLO}} - \mathcal{B}_{\text{NLO}}^{3\text{L}\to 0}}{\mathcal{B}_{\text{LO}}} = \frac{\alpha_s(\mu_b)}{\pi} \frac{C_7^{(1)\text{eff}}(\mu_b) - \left[C_7^{(1)\text{eff}}(\mu_b)\right]^{3\text{L}\to 0}}{2\,C_7^{(0)\text{eff}}(\mu_b)}, \tag{6.7}$$

and

$$\frac{C_7^{(1)\text{eff}}(\mu_b) - \left[C_7^{(1)\text{eff}}(\mu_b)\right]^{3\text{L}\to 0}}{2\,C_7^{(0)\text{eff}}(\mu_b)} = \frac{h_1^{(1)}\eta^{a_1+1} + h_2^{(1)}\eta^{a_2+1} + \sum_{i=3}^8 h_i^{(1)}\eta^{a_i}}{\eta^{a_2}C_7^{(0)}(\mu_0) + \frac{8}{3}\left(\eta^{a_1} - \eta^{a_2}\right)C_8^{(0)}(\mu_0) + \sum_{i=1}^8 h_i^{(0)}\eta^{a_i}}. \tag{6.8}$$

The ratios (6.7) and (6.8) are shown in figure 3 as functions of η and μ_b.

Comparing the right plots in figures 2 and 3, one can see that the two corrections are similar in magnitude. The suppression of the NNLO one by an additional factor of $\alpha_s(\mu_b)/\pi$ tends to get compensated by larger values of $h_i^{(2)}$ as compared to $h_i^{(1)}$. However, as the remaining plots show, the ratios (6.6) and (6.8) do not contradict the naive expectation of being "numbers of order unity". Thus, there seems to be no deep reason behind the observed relative smallness of the NLO effect and the relatively large value of the NNLO one. It should be remembered that the other NLO corrections exceed $\sim 30\%$ while the other NNLO ones stay within $\sim 10\%$ [3].

It is evident from figure 2 that the μ_b-dependence of the NNLO correction (6.1) for $\mu_b \in [1, 10]\,\text{GeV}$ originates almost entirely from the overall factor $\alpha_s^2(\mu_b)$. The $\pm 3\%$ higher-order uncertainty in $\mathcal{B}_{\text{NNLO}}$ that was estimated in refs. [3, 4] takes this μ_b-dependence into account. The central value of $\mathcal{B}_{\text{NNLO}}$ was calculated there for $\mu_b = 2.5\,\text{GeV}$. At that scale, our present correction amounts to around -4.4%. It is worth noting that $\mathcal{B}_{\text{NNLO}}^{4\text{L}\to 0}$ calculated along the formulae of ref. [4] turns out to be much less μ_b-dependent than the more complete $\mathcal{B}_{\text{NNLO}}$.

The actual numerical calculation in refs. [3, 4] included contributions from $B_{i7}^{(2)}$ but not from the very recently found $B_{i8}^{(2)}$. This fact is practically irrelevant for the above discussion. The numerical effect of $B_{i8}^{(2)}$ on $\mathcal{B}_{\text{NNLO}}$ is about 10 times smaller than the one of $B_{i7}^{(2)}$, and has the same sign. The small contribution of $B_{i8}^{(2)}$ will be taken into account together with other similar ones in a future upgrade of the phenomenological NNLO analysis.

7. Summary

We have evaluated the complete four-loop anomalous dimension matrix $B^{(2)}$ that is necessary for determining the effective FCNC couplings $\bar{q}q'\gamma$ and $\bar{q}q'g$ at the NNLO in QCD. The results are presented in a form that applies to any external flavour case, with an arbitrary number of non-decoupled quarks. Adding an essential contribution to the NNLO QCD analysis $\bar{B} \to X_s\gamma$ has been the main purpose of our calculation. The obtained $\mathcal{O}(\alpha_s^2(\mu_b))$ correction to the branching ratio of this decay amounts to around -2.9% for $\mu_b = 5\,\text{GeV}$, and -4.4% for $\mu_b = 2.5\,\text{GeV}$.

k	l		
	$n=0$	$n=1$	$n=2$
1,2	1,2	2	none
3,4,5,6	1,2,4	2	none
7	all	1,2,4,7,8	2,7,8
8	1,2,4,8	2,8	none

Table 2: Indices k and l of the relevant $U_{kl}^{(n)}$ for $n = 0, 1, 2$.

Acknowledgments

A. Solutions to the RGE

This appendix contains solutions to the RGEs for all the Wilson coefficients that matter for $b \to s\gamma$ at the NNLO before including higher-order electroweak corrections. A generic solution to the RGEs (2.4) reads

$$\vec{C}^{\text{eff}}(\mu_b) = \hat{U}(\mu_b, \mu_0)\vec{C}^{\text{eff}}(\mu_0). \tag{A.1}$$

The block-triangular structure of the matrix $\hat{U}$ corresponds to that of $\left(\hat{\gamma}^{\text{eff}}\right)^T$, taking into account that the blocks $A^{(n)}$ and $C^{(n)}$ in eq. (2.6) have a block-triangular form, too. In particular, $U_{87} = 0$, and $U_{kl} = 0$ for $k \leq 2$ and $l > 2$.

Using the perturbative expansions for $C_i^{\text{eff}}(\mu)$ (6.2) and

$$\hat{U}(\mu_b, \mu_0) = \sum_{n \geq 0} \widetilde{\alpha}^n(\mu_0)\hat{U}^{(n)}, \tag{A.2}$$

one easily finds that

$$\vec{C}^{(0)\text{eff}}(\mu_b) = \hat{U}^{(0)}\vec{C}^{(0)\text{eff}}(\mu_0), \tag{A.3}$$

$$\vec{C}^{(1)\text{eff}}(\mu_b) = \eta \left[\hat{U}^{(0)}\vec{C}^{(1)\text{eff}}(\mu_0) + \hat{U}^{(1)}\vec{C}^{(0)\text{eff}}(\mu_0)\right], \tag{A.4}$$

$$\vec{C}^{(2)\text{eff}}(\mu_b) = \eta^2 \left[\hat{U}^{(0)}\vec{C}^{(2)\text{eff}}(\mu_0) + \hat{U}^{(1)}\vec{C}^{(1)\text{eff}}(\mu_0) + \hat{U}^{(2)}\vec{C}^{(0)\text{eff}}(\mu_0)\right]. \tag{A.5}$$

The matrices $\hat{U}^{(n)}$ are functions of η only[3]

$$U_{kl}^{(n)} = \sum_{j=0}^{n} \sum_{i=1}^{8} m_{kli}^{(nj)} \eta^{a_i - j}. \tag{A.6}$$

The powers a_i have been given in table 1. Only a relatively small set of non-vanishing $U_{kl}^{(n)}$ is relevant for $b \to s\gamma$ at the NNLO, so long as

$$C_l^{(0)\text{eff}}(\mu_0) = 0, \quad \text{for } l = 1, 3, 4, 5, 6, \tag{A.7}$$

$$C_l^{(1)\text{eff}}(\mu_0) = 0, \quad \text{for } l = 3, 5, 6, \tag{A.8}$$

which is the case in the SM. The corresponding values of the indices k and l for $n = 0, 1, 2$ are collected in table 2. The "magic numbers" $m_{kli}^{(nj)}$ that occur in the expressions (A.6) for these $U_{kl}^{(n)}$ are given in tables 3–5.

As far as $C_i^{(n)\text{eff}}(\mu_0)$ are concerned, their SM values for $i = 1, \ldots, 6$ can be found in section 2 of ref. [16]. The coefficients $C_i^{Q(n)}$ from that paper combine with a relative minus sign to our $C_i^{(n)}$, i.e. $C_i^{(n)} = C_i^{t(n)} - C_i^{c(n)}$. For $i = 7, 8$, one can find the corresponding $C_i^{Q(n)}$ in section 6 of ref. [17]. They combine to our $C_i^{(n)}$ according to the relation $C_i^{(n)} = C_i^{t(n+1)} - C_i^{c(n+1)}$. The upper index gets shifted because Q_7 and Q_8 in ref. [17] were normalized as X_7 and X_8 in eq. (5.2) here. Once $C_i^{(n)}(\mu_0)$ are evaluated, eq. (2.3) should be applied to obtain $C_i^{(n)\text{eff}}(\mu_0)$.

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[3]The expressions for the matrices $\gamma^{(n)\text{eff}}$ and $U^{(n)}$ in Mathematica format can be found under http://www.fuw.edu.pl/~misiak/hep-ph.0612329.math/hep-ph.0612329.math.txt.

i	1	2	3	4	5	6	7	8
$m_{11i}^{(00)}$	0	0	0.3333	0.6667	0	0	0	0
$m_{12i}^{(00)}$	0	0	1	-1	0	0	0	0
$m_{21i}^{(00)}$	0	0	0.2222	-0.2222	0	0	0	0
$m_{22i}^{(00)}$	0	0	0.6667	0.3333	0	0	0	0
$m_{31i}^{(00)}$	0	0	0.0106	0.0247	-0.0129	-0.0497	0.0092	0.0182
$m_{32i}^{(00)}$	0	0	0.0317	-0.0370	-0.0659	0.0595	-0.0218	0.0335
$m_{34i}^{(00)}$	0	0	0	0	-0.1933	0.1579	0.1428	-0.1074
$m_{41i}^{(00)}$	0	0	0.0159	-0.0741	0.0046	0.0144	0.0562	-0.0171
$m_{42i}^{(00)}$	0	0	0.0476	0.1111	0.0237	-0.0173	-0.1336	-0.0316
$m_{44i}^{(00)}$	0	0	0	0	0.0695	-0.0459	0.8752	0.1012
$m_{51i}^{(00)}$	0	0	-0.0026	-0.0062	0.0018	0.0083	-0.0004	-0.0009
$m_{52i}^{(00)}$	0	0	-0.0079	0.0093	0.0094	-0.0100	0.0010	-0.0017
$m_{54i}^{(00)}$	0	0	0	0	0.0274	-0.0264	-0.0064	0.0055
$m_{61i}^{(00)}$	0	0	-0.0040	0.0185	0.0021	-0.0136	-0.0043	0.0012
$m_{62i}^{(00)}$	0	0	-0.0119	-0.0278	0.0108	0.0163	0.0103	0.0023
$m_{64i}^{(00)}$	0	0	0	0	0.0317	0.0432	-0.0675	-0.0074
$m_{71i}^{(00)}$	0.5784	-0.3921	-0.1429	0.0476	-0.1275	0.0317	0.0078	-0.0031
$m_{72i}^{(00)}$	2.2996	-1.0880	-0.4286	-0.0714	-0.6494	-0.0380	-0.0185	-0.0057
$m_{73i}^{(00)}$	8.0780	-5.2777	0	0	-2.8536	0.1281	0.1495	-0.2244
$m_{74i}^{(00)}$	5.7064	-3.8412	0	0	-1.9043	-0.1008	0.1216	0.0183
$m_{75i}^{(00)}$	202.9010	-149.4668	0	0	-55.2813	2.6494	0.7191	-1.5213
$m_{76i}^{(00)}$	86.4618	-59.6604	0	0	-25.4430	-1.2894	0.0228	-0.0917
$m_{77i}^{(00)}$	0	1	0	0	0	0	0	0
$m_{78i}^{(00)}$	2.6667	-2.6667	0	0	0	0	0	0
$m_{81i}^{(00)}$	0.2169	0	0	0	-0.1793	-0.0730	0.0240	0.0113
$m_{82i}^{(00)}$	0.8623	0	0	0	-0.9135	0.0873	-0.0571	0.0209
$m_{84i}^{(00)}$	2.1399	0	0	0	-2.6788	0.2318	0.3741	-0.0670
$m_{88i}^{(00)}$	1	0	0	0	0	0	0	0

Table 3: "Magic numbers" for the relevant $U_{kl}^{(0)}$.

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i	1	2	3	4	5	6	7	8
$m_{12i}^{(10)}$	0	0	-2.9606	-4.0951	0	0	0	0
$m_{12i}^{(11)}$	0	0	5.9606	1.0951	0	0	0	0
$m_{22i}^{(10)}$	0	0	-1.9737	1.3650	0	0	0	0
$m_{22i}^{(11)}$	0	0	1.9737	-1.3650	0	0	0	0
$m_{32i}^{(10)}$	0	0	-0.0940	-0.1517	-0.2327	0.2288	0.1455	-0.4760
$m_{32i}^{(11)}$	0	0	-0.5409	1.6332	1.6406	-1.6702	-0.2576	-0.2250
$m_{42i}^{(10)}$	0	0	-0.1410	0.4550	0.0836	-0.0664	0.8919	0.4485
$m_{42i}^{(11)}$	0	0	2.2203	2.0265	-4.1830	-0.7135	-1.8215	0.7996
$m_{52i}^{(10)}$	0	0	0.0235	0.0379	0.0330	-0.0383	-0.0066	0.0242
$m_{52i}^{(11)}$	0	0	0.0400	-0.1861	-0.1669	0.1887	0.0201	0.0304
$m_{62i}^{(10)}$	0	0	0.0352	-0.1138	0.0382	0.0625	-0.0688	-0.0327
$m_{62i}^{(11)}$	0	0	-0.2614	-0.1918	0.4197	0.0295	0.1474	-0.0640
$m_{71i}^{(10)}$	0.0021	-1.4498	0.8515	0.0521	0.6707	0.1220	-0.0578	0.0355
$m_{71i}^{(11)}$	-4.3519	3.0646	1.5169	-0.5013	0.3934	-0.6245	0.2268	0.0496
$m_{72i}^{(10)}$	9.9372	-7.4878	1.2688	-0.2925	-2.2923	-0.1461	0.1239	0.0812
$m_{72i}^{(11)}$	-17.3023	8.5027	4.5508	0.7519	2.0040	0.7476	-0.5385	0.0914
$m_{74i}^{(10)}$	-8.6840	8.5586	0	0	0.7579	0.4446	0.3093	0.4318
$m_{74i}^{(11)}$	-42.9356	30.0198	0	0	5.8768	1.9845	3.5291	-0.2929
$m_{77i}^{(10)}$	0	7.8152	0	0	0	0	0	0
$m_{77i}^{(11)}$	0	-7.8152	0	0	0	0	0	0
$m_{78i}^{(10)}$	17.9842	-18.7604	0	0	0	0	0	0
$m_{78i}^{(11)}$	-20.0642	20.8404	0	0	0	0	0	0
$m_{82i}^{(10)}$	3.7264	0	0	0	-3.2247	0.3359	0.3812	-0.2968
$m_{82i}^{(11)}$	-5.8157	0	1.4062	-3.9895	3.2850	3.6851	-0.1424	0.6492
$m_{88i}^{(10)}$	6.7441	0	0	0	0	0	0	0
$m_{88i}^{(11)}$	-6.7441	0	0	0	0	0	0	0

Table 4: "Magic numbers" for the relevant $U_{kl}^{(1)}$.

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i	1	2	3	4	5	6	7	8
$m_{72i}^{(20)}$	-212.4136	167.6577	5.7465	-3.7262	28.8574	-2.1262	2.2903	0.1462
$m_{72i}^{(21)}$	-74.7681	58.5182	-13.4731	3.0791	7.0744	2.8757	3.5962	-1.2982
$m_{72i}^{(22)}$	31.4443	-18.1165	23.2117	13.2771	-19.8699	4.0279	-8.6259	2.6149
$m_{77i}^{(20)}$	0	44.4252	0	0	0	0	0	0
$m_{77i}^{(21)}$	0	-61.0768	0	0	0	0	0	0
$m_{77i}^{(22)}$	0	16.6516	0	0	0	0	0	0
$m_{78i}^{(20)}$	15.4051	-18.7662	0	0	0	0	0	0
$m_{78i}^{(21)}$	-135.3141	146.6159	0	0	0	0	0	0
$m_{78i}^{(22)}$	36.4636	-44.4043	0	0	0	0	0	0

Table 5: "Magic numbers" for the relevant $U_{kl}^{(2)}$.

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